Analysis on Surreal Numbers

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1. Created by J.H. Conway in 1976 in his book *On Numbers and Games* – written in a single week.
2. Popularized by Knuth in his novella *Surreal Numbers: How Two Ex-Students Turned to Pure Mathematics and Found Total Happiness*.

Goal: To construct the largest possible ordered "field", which we shall call **No**, for numbers.

Finite Days

$$0 = \{\varnothing \mid \varnothing\} = \{ \mid \}$$

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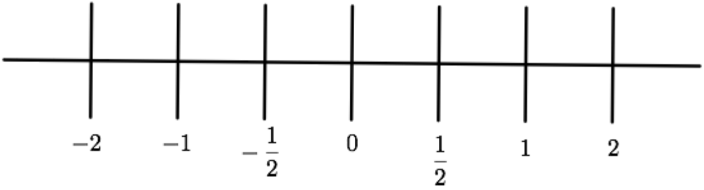
$$1 = \{0 \mid\ \}$$
$$-1 = \{\ \mid 0\}$$



$$0 = \{ \mid \}$$

$$1 = \{0 \mid \} \qquad 2 = \{0, 1 \mid \}$$

$$-1 = \{ \mid 0\} \qquad -2 = \{ \mid -1, 0\}$$

$$\frac{1}{2} = \{0 \mid 1\} \qquad \frac{-1}{2} = \{-1 \mid 0\}$$

Day ω

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots=\sum_{n=1}^{\infty}\frac{1}{4^n}=\frac{1/4}{1-\frac{1}{4}}=\frac{1}{3}$$

$$\frac{1}{2}-\frac{1}{8}-\frac{1}{32}-\cdots=\frac{1}{2}-\sum_{n=1}^{\infty}\frac{1}{2^{2n+1}}=\frac{1}{2}-\frac{1/8}{1-\frac{1}{4}}=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$$

Then we should have
$\frac{1}{3}=\left\{\frac{1}{4},\frac{1}{4}+\frac{1}{16},\frac{1}{4}+\frac{1}{16}+\frac{1}{64}\mid\frac{1}{2},\frac{1}{2}-\frac{1}{8},\frac{1}{2}-\frac{1}{8}-\frac{1}{32}\right\}$, as one can check.

Furthermore on day ω, we get...

$$\omega = \{0, 1, 2, 3, \cdots \mid \}$$
$$-\omega = \{ \mid -1, -2, -3, -4, \cdots\}$$
$$\epsilon = \left\{0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots \right\}$$

On future days...

$$\omega - 1 = \{0, 1, 2, 3, \cdots \mid \omega\}$$
$$\frac{\omega}{2} = \{0, 1, 2, 3, \cdots \mid \omega - 1, \omega - 2\omega - 3, \cdots\}$$
$$\sqrt{\omega} = \left\{0, 1, 2, 3, \cdots \mid \omega, \frac{\omega}{2}, \frac{\omega}{3}, \cdots \right\}$$

Here will go gaps

Definition (Ordering)

We say $x \geq y$ if and only if the following holds:

(i) there is no x^R such that $x^R \leq y$.

(ii) there is no y^L such that $x \leq y^L$.

We say that $x \leq y$ if and only if $y \geq x$. Furthermore, we define $x = y$ if and only if $x \geq y$ and $y \leq x$. Finally, we define $x \geq y$ if and only if $x \geq y$ and $y \not\geq x$ - $x < y$ is defined mutatis mutandis.

Let $x, y \in \mathbf{No}$.

$$x + y = \left\{x^L + y, x + y^L \mid x^R + y, x + y^R\right\}$$

$$-x = \left\{-x^R \mid -x^L\right\}$$

$$\begin{aligned} xy =& \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R \mid \\ & x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\} \end{aligned}$$

Given $xy = 1$, we have

$$\begin{aligned} y =& \Big\{0, \frac{1 + (x^R - x)y^L}{x^R}, \frac{1 + (x^L - x)y^R}{x^L} \Big| \\ & \frac{1 + (x^L - x)y^L}{x^L}, \frac{1 + (x^R - x)y^R}{x^R}\Big\} \end{aligned}$$

Definition (Dedekind Representation)

For $x \in \mathbf{No}^{\mathcal{D}}$*, the Dedekind representation of* x *is* $x = \{\mathbf{No}_{<x} \mid \mathbf{No}_{>x}\}$.

Remark

All the previous defined algebraic structure on **No** holds for $\mathbf{No}^{\mathcal{D}}$.

Definition (Genetic Functions)

$$\left\{ \bigcup_{x^L \in L_x, x^R \in R_x} \{f^L(x) : f^L \in L_f(x^L, x^R)\} \right| \left. \bigcup_{x^L \in L_x, x^R \in R_x} \{f^R(x) : f^R \in R_f(x^L, x^R)\} \right\}$$

Example

The function $f(x) = x^2$ is represented as $f(x) = \left\{ 2xx^L - x^{L^2}, 2xx^R - x^{R^2} \mid xx^L + xx^R - x^L x^R \right\}$.

Example (Arctan)

Let $[x]_n$ be the n-truncation of the Maclaurin expansion of $f(x)$, i.e. $[x]_n = \sum_{i=0}^{n} \frac{f^(i)(0)}{i!} x^i$. Then for all **No**, we define

$$\underline{\text{arctan}}(x) = \left\{ \frac{-\pi}{2}, \underline{\text{arctan}}(x^L) + \left[\frac{x - x^L}{1 + xx^L}\right]_{4n-1}, \underline{\text{arctan}}(x^R) + \left[\frac{x - x^R}{1 + xx^R}\right]_{4n+1} \middle| \underline{\text{arctan}}(x^R) - \left[\frac{x^R - x}{1 + xx^R}\right]_{4n-1}, \underline{\text{arctan}}(x^L) - \left[\frac{x^L - x}{1 + xx^L}\right]_{4n+1}, \frac{\pi}{2} \right\}$$

Definition (Topology)

A topology on **No** *is a collection $\mathcal{A}$ of subclasses of* **No** *satisfying the following properties:*

1. $\emptyset, \mathbf{No} \in \mathcal{A}$
2. $\bigcup_{i\in I} A_i \in \mathcal{A}$ *for any subcollection* $\{A_i\}_{i\in I} \subset \mathcal{A}$ *indexed over a proper set* I*.*
3. $\bigcap_{i\in I} A_i \in \mathcal{A}$ *for any subcollection* $\{A_i\}_{i\in I} \subset \mathcal{A}$ *indexed over a finite set* I*.*

We call the elements of $\mathcal{A}$ open sets.

Definition (**No** Topology)

We declare $\emptyset$ to be open. A nonempty subinterval of **No** is open if

1. its endpoints are in $\mathbf{No} \cup \{\mathbf{On}, \mathbf{Off}\}$
2. it does not contain its endpoints

A subclass $A \subset \mathbf{No}$ is said to be open if it has the form $A = \bigcup_{i \in I} A_i$, where I is a proper set and the A_i are open intervals.

Definition (Continuous)

Let $A \subset \mathbf{No}$ and $f : A \to \mathbf{No}$ be a function. We say that f is continuous on A if for any open class $B \subset \mathbf{No}$, $f^{-1}(A)$ is open in A [in the Subspace Topology].

Definition (Limits)

Let $\mathcal{U} = a_1, a_2, \cdots$ be an **On***-length sequence. Define*

$$\ell(\mathcal{U}) = \left\{ a : a < \sup \left(\bigcap_{i \geq 1} \bigcap_{j \geq i} \mathscr{L}_{a_j} \right) \middle| b : b > \inf \left(\bigcap_{i \geq 1} \bigcap_{j \geq i} \mathscr{R}_{a_j} \right) \right\}$$

Definition

Let $\mathcal{U} = a_1, a_2, \cdots$ be an **On***-length sequence. We say the limit of $\mathcal{U}$ is ℓ and write $\lim_{i \to \mathbf{On}} a_i = \ell$ if the expression on the right of the above definition is a Dedekind representation and $\ell = \ell(\mathcal{U})$.*

Example

Consider the following **On**-length sequence:

$$\mathcal{U} = 1,\ \ 1 + \frac{1}{\omega},\ \ 1 + \frac{1}{\omega} + \frac{1}{\omega^2},\ \ 1 + \frac{1}{\omega} + \frac{1}{\omega^2} + \frac{1}{\omega^3} \cdots$$

There does indeed exist $N \in \mathbf{On}$ such that for $n, m \in \mathbf{On}_{>N}$

$$\left| \sum_{i \in \mathbf{On}_{\leq m}} \frac{1}{\omega^i} - \sum_{i \in \mathbf{On}_{\leq n}} \frac{1}{\omega^i} \right| < \epsilon$$

However, the Dedekind representation for $\ell(\mathcal{U})$ is $\sum_{i \in \mathbf{On}} \frac{1}{\omega^i}$, which is a gap. So $\mathcal{U}$ is a non-convergent Cauchy sequence. Therefore, **No** is not complete.

Definition (Limits of Functions)

Let f be a function defined everywhere on an open neighborhood of x_0 except possibly at x_0. We say that f converges to ℓ as x converges to x_0, denoted $\lim_{x \to x_0} f(x) = \ell$ if the following expression is a Dedekind representation for ℓ:

$$\left\{ p : \left(p < \sup \bigcup_{b < x < x_0} \bigcap_{x \le y < x_0} \mathscr{L}_{f(y)} \right) \right| \\ q : q > \left(\inf \bigcup_{x_0 < x < c} \bigcap_{x_0 < y \le x} \mathscr{R}_{f(y)} \right) \right\}$$

Remark

The Dedekind representation notion of a limit is equivalent to the $\epsilon - \delta$ definition of a limit.

Definition (Connected)

A class $T \subset \mathbf{No}$ is connected if there does not exist a separation of T; that is, there does not exist a pair of nonempty disjoint classes U, V, open in T, such that $T = U \cup V$.

Remark

Every convex class $T \subset \mathbf{No}$ is connected.

Theorem

If f is continuous on $[a, b]$, then the image of $f([a, b])$ is connected.

Theorem (Intermediate Value Theorem)

If f is continuous on $[a, b] \subset \mathbf{No}$, then for every $u \in \mathbf{No}$ that lies between $f(a)$ and $f(b)$, there exists a number $p \in [a, b]$ such that $f(p) = u$.

Conjecture

The only functions reaching a number at a gap are constant on an open interval containing the gap.

Definition (**No** Metric)

If $\mathbf{a}, \mathbf{b} \in \mathbf{No}^2$, we define $d(a, b)$ to be the "normal Euclidean metric".

Definition

If *abcd* is a rectangle in $\mathbf{No}^2$, we define the area of *abcd* to be $d(a, b)d(c, d)$.

Definition (Riemann Sum)

Let f be a continuous function on $[a, b]$ except possibly at finitely many points. Suppose that for all $n \in \mathbb{N}$ and $c, d \in [a, b]$ with $c \leq d$. Define

$$g(n, c, d) = \sum_{i=0}^{n} \frac{d-c}{n} f\left(c + i\left(\frac{d-c}{n}\right)\right)$$

where $g \in \overline{K}'$. Then for all $\alpha \in \mathbf{On}$, define the αth Riemann sum of f on $[a, b]$ to be $g(\alpha, a, b)$.

Theorem (Extreme Value Theorem)

Let $A \subset \mathbf{No}$ be a strongly compact and bounded and $f : A \to \mathbf{No}$ be strongly continuous and bounded. Then there exists $c, d \in A$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in A$.

Theorem

If f is strongly continuous and bounded on $[a, b] \subset$ **No***, then the function g defined for all $x \in [a, b]$ by $g(x) = \int_a^x f(t)\, dt$ is weakly continuous on $[a, b]$ and satisfies $g'(x) = f(x)$ for all $x \in (a, b)$.*

Conjecture

Let $f : A \to \mathbf{No}$ *be a genetic function defined on a locally open subinterval* $A \subset \mathbf{No}$*. If there exists a genetic function* $F : A \to \mathbf{No}$ *such that* $F'(x) = f(x)$ *for all* $x \in A$*, then* F *is unique up to additive constant.*

unctions: There is a lack of complete analogies between real functions and surreal functions, especially those important for a rich theory of analysis – sin, cos, et cetera. Importantly, one would despite a genetic formula for Dedekind representation for the definite integral of a function.

Series: There is no method for evaluating series – at least in any general sense – on the surreals. Specifically, there is no surreal version of evaluating series which does not depend on the *n*th partial sum. This would allow for the development of series tests and more importantly a concept of power series for functions. If one were to generalize the standard analysis functions as above, then one could develop a theory of Fourier Series for surreal functions.

egration: The Fundamental Theorem of Calculus does not yet hold in the way we would like – unique primitives up to additive constant. It remains open to show that this can hold for the integral defined above in the conditions we have considered or to create an integral which satisfies these properties.

entiation: There has yet to be a satisfactory definition of a surreal derivative. This would allow for a theory of differential