Torsion of Rational Elliptic Curves over Nonic Galois Fields

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Definition (Elliptic Curve)

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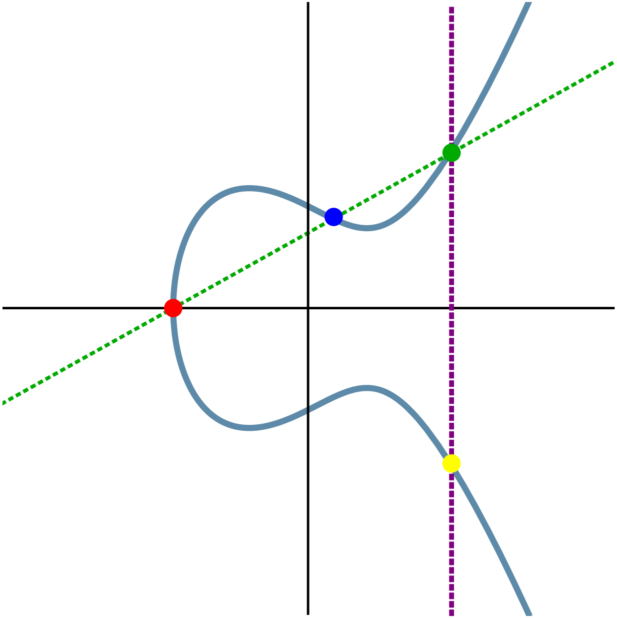
$$E = \{(x, y) : y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}$$



Theorem (Mordell-Weil)

For any abelian variety A over a number field K, $A(K)$ is a finitely generated abelian group, i.e.

$$A(K) \cong \mathbf{Z}^r \oplus \mathrm{Tor}(A(K))$$

$$E(K)_{\text{tors}} \cong {}^{\mathbf{Z}}\!/_{m\mathbf{Z}} \oplus {}^{\mathbf{Z}}\!/_{mn\mathbf{Z}}$$

$$E[m] \cong {}^{\mathbf{Z}}\!/_{m\mathbf{Z}} \oplus {}^{\mathbf{Z}}\!/_{m\mathbf{Z}}$$

Theorem (Merel, 1994)

For all $d \in \mathbf{Z}_+$*, there exists a constant* $B(d) \geq 0$ *such that for all elliptic curves* E *over a number field* K *with* $[K\colon \mathbf{Q}] = d$*, then*

$$|E(K)_{tors}| \leq B(d).$$

Theorem (Merel, 1994)

Let E/K be an elliptic curve with $[K\colon \mathbf{Q}] = d > 1$ and p be prime. If $E(K)$ has a p-torsion point, then $p < d^{3d^2}$.

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This was later improved by Oesterlé to $(3^{d/2} + 1)^2$. In the case over $\mathbf{Q}$, Lozano-Robledo improves this to $2d + 1$, $p \geq 11$ and $p \neq 13, 37$.

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$$S_{\mathbf{Q}}(d) \subset S(d)$$

Theorem (Mazur, Parent, Derickx, Kammienny, Stein, Stoll, Lozano-Robledo, et al.)

$$S_{\mathbf{Q}}(9) = \{2, 3, 5, 7, 11, 13, 17, 19\}$$
$$S_{\mathbf{Q}}(21) = \{2, 3, 5, 7, 11, 13, 17, 19, 37, 43\}$$

$$\Phi(d) := \{\text{Set of Isomorphism Classes of } E(K)_{\text{tors}} \colon [K \colon \mathbf{Q}] = d\}$$

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$$\Phi_{\mathbf{Q}}(d) \subseteq \Phi(d)$$

Theorem (Mazur)

$\Phi(1) = \{C_n : n = 1, \ldots, 10, 12\} \cup \{C_2 \times C_{2n} : n = 1, \ldots, 4\}$

Theorem (Najman, González-Jiménez, et al.)

$$\Phi_{\mathbf{Q}}(2) = \{C_n \colon n = 1, \ldots, 10, 12, 15, 16\} \cup \{C_2 \times C_{2n} \colon n = 1, \ldots, 6\}$$

$$\Phi_{\mathbf{Q}}(3) = \{C_n \colon 1, \ldots, 10, 12, 13, 14, 18, 21\} \cup \{C_2 \times C_{2n} \colon n, 1, 2, 3, 4, 7\}$$

$$\Phi_{\mathbf{Q}}(4) = \{C_n \colon n = 1, \ldots, 10, 12, 13, 15, 16, 20, 24\} \cup \{C_2 \times 2_{2n} \colon n = 1, \ldots, 6, 8\} \cup \{C_3 \times C_{3n} \colon n = 1, 2\} \cup \{C_4 \times C_{4n} \colon n = 1, 2\} \cup \{C_5 \times C_5\} \cup \{C_6 \times C_6\}$$

$$\Phi_{\mathbf{Q}}(5) = \{C_n \colon n = 1, \ldots, 12, 25\} \cup \{C_2 \times C_{2n} \colon n = 1, \ldots, 4\}$$

Definition (Isogeny)

Let E_1, E_2 be elliptic curves. An isogeny from E_1 to E_2 is a morphism $\phi : E_1 \to E_2$ with $\phi(\mathcal{O}) = \mathcal{O}$.

Theorem (Fricke, Kenku, Klein, Kubert, Ligozat, Mazur, Ogg, et al.)

If $E/\mathbf{Q}$ *has an* n*-isogeny over* $\mathbf{Q}$*, then* $n \leq 19$ *or* $n \in \{21, 25, 27, 37, 43, 67, 163\}$*. If* E *does not have CM, then* $n \leq 18$ *or* $n \in \{21, 25, 37\}$*.*

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- $\overline{\rho}_{E,n} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}(E[n]) \cong \mathrm{GL}_2(\mathbf{Z} / n\mathbf{Z})$

Possible images and indices of Galois representations are limited by the work of Zureick-Brown, Clark, Zywina, Rouse, Corn, Rice, Stankewicz, et al.

Definition (Weil Pairing)

If E/K be an elliptic curve and $m \geq 2$, then there exists a bilinear, nondegenerate, alternating, Galois invariant pairing on $E[m]$.

Idea

Bound the torsion subgroup (by some large sum of Sylow subgroups), then eliminate cases by isogeny, Galois representations, and the Weil pairing.

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$$\begin{aligned}
[K \colon \mathbf{Q}(\zeta_n)]\,[\mathbf{Q}(\zeta_n) \colon \mathbf{Q}] &= [K \colon \mathbf{Q}] \\
[K \colon \mathbf{Q}(\zeta_n)]\phi(n) &= [K \colon \mathbf{Q}]
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but $\phi(n)$ is even for $n > 2$, a contradiction. □

Now K cannot contain full 7-torsion so that $E(K)[7^\infty] = \mathbf{Z}\,/\,7^m\mathbf{Z}$. In particular, $E/\mathbf{Q}$ has a 7^k-isogeny. But $7^k \geq 49$ for $k > 1$, which is not a possible isogeny. Therefore,

$$E(K)[7^\infty] \subseteq {}^{\mathbf{Z}}\!/_{7\mathbf{Z}}$$

Theorem

If $K/\mathbf{Q}$ is a Galois extension of degree 9 and $E/\mathbf{Q}$ is an elliptic curve, then

$$E(K)_{tors} \subseteq \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/6983776800\mathbf{Z}$$

Furthermore, there are at most 34 possibilities for $E(K)_{tors}$.